GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

ONE DIMENSIONAL FUZZY WAVE EQUATION WITH LAPLACE TRANSFORMATION

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ABSTRACT

In this article we study the method of solving one dimensional fuzzy wave equation under certain condition by using fuzzy Laplace transformation. Finally we give some illustrative examples.

Keywords: Fuzzy numbers, Fuzzy partial differential equation, Fuzzy Laplace Transformation.

1. INTRODUCTION

The concept of fuzzy sets and set operations was first introduced by Zadeh [23] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Seikhala defined fuzzy derivatives while concept of integration of fuzzy functions was first introduced by Dubois and Prade [3], later on studied by Matloka [7], Mordeson and Newman [6] and many others. The idea of fuzzy partial differential equations was first introduced by Buckley [5]. Allahveranloo [8] proposed the difference method for solving fuzzy partial differential equations. The technique of solving first order differential equations by using fuzzy Laplace Transformation was proposed by Allahveranloo and Ahmadi [9] under generalized H- differentiability.

2. DEFINITION AND BACKGROUND

A fuzzy number is a fuzzy subset of the real line *R* i.e a function $u : R \to [0,1]$ which is bounded, convex and normal. Let *E* denote the set of all fuzzy numbers which are upper semi continuous and have compact support. The - level set set of a fuzzy real number *u* for 0 < 1, defined as = { *t* *R* : *u*(*t*) }.

A fuzzy real number *u* is called *convex*, if *u*(*t*) *u*(*s*) *u*(*r*) = min (*u*(*s*), *u*(*r*)), where *s* < *t* < *r*.

If there exists t_0 *R* such that *u*(t_0) = 1, then the fuzzy real number *u* is called *normal*.

A fuzzy real number *u* is said to be *upper semi- continuous* if for each > 0, u^{-1}([0, *a* +)), for all *a* *[0,1]* is open in the usual topology of *R*.

The absolute value |*u*| of *u* *E* is defined as :

$$|u|(t) = \max \{ u(t), u(-t) \}, \quad \text{if } t \geq 0$$
$$= 0, \quad \text{if } t < 0 .$$

It is clear that that the - level set of the fuzzy number u is a closed and bounded interval where $\underline{u}(\alpha)$ denotes the left-hand end point and denotes the right-hand end point ofTwo arbitrary fuzzy numbers and are said to be equal i.e if and only if and . Since each can be regarded as a fuzzy number defined by :

The Hausdorff distance between fuzzy numbers is a mapping $\overline{d} : L(R) \times L(R) \to R_+$ defined by

$$d(u,v) = \sup_{0 \le \alpha \le 1} \max \left\{ \left| \underline{u}(\alpha) - \underline{v}(\alpha) \right|, \left| \overline{u}(\alpha) - \overline{v}(\alpha) \right| \right\}$$

Where and . It can be easily shown that d is a metric on $L(R)$ with the following properties :

1. $d(u+w, v+w) = d(u,v)$ for all $u,v,w \in L(R)$.
2. $d(ku, kv) = |k|\, d(u,v)$ for all $u,v \in L(R)$
3. $d(u+v, w+e) \le d(u,w) + d(v,e)$ for all $u,v,w,e \in L(R)$
4. $(d, L(R))$ is a complete metric space.

Definition2.1: Let $f : R \to L(R)$ be a fuzzy valued function. f is said to be continuous at $t_0 \in R$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(f(t), f(t_0)) < \varepsilon \quad \text{whenever} \quad |t - t_0| < \delta .$$

Definition2.2: Let $f : R \to L(R)$ be a fuzzy valued function and $x_0 \in R$. We say that f is differentiable at x_0, if there exists $f'(x_0) \in L(R)$ such that

$$\text{(a)} \quad \lim_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0+} \frac{f(x_0) - f(x_0-h)}{h} = f'(x_0)$$

or

$$\text{(b)} \quad \lim_{h \to 0-} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0-} \frac{f(x_0) - f(x_0-h)}{h} = f'(x_0)$$

Theorem 2.1: Let $f : R \to L(R)$ be a fuzzy valued function and denote $f(t) = \left[\underline{f}(t,\alpha), \overline{f}(t,\alpha)\right]$ for each $0 \le \alpha \le 1$ followings are hold –

(a) If f is differentiable in the first form (a) in definition 2.2 , then $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are differentiable and $f'(t) = \left[\underline{f}'(t,\alpha), \overline{f}'(t,\alpha)\right]$.

(b) If f is differentiable in the 2nd form (b) in definition 2.2 , then $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are differentiable and $f'(t) = \left[\overline{f}'(t,\alpha), \underline{f}'(t,\alpha)\right]$.

Theorem 2.2: Let $f : R \to L(R)$ be a fuzzy valued function and denote $f(t) = \left[\underline{f}(t,\alpha), \overline{f}(t,\alpha)\right]$ for each $0 \le \alpha \le 1$, Then

(a) If f and f' are differentiable in the first form (a) in definition 2.2 or if f and f' are differentiable in the 2nd form (b) in definition 2.2 , then $\underline{f}'(t,\alpha)$ and $\overline{f}'(t,\alpha)$ are differentiable and $f''(t) = \left[\underline{f}''(t,\alpha), \overline{f}''(t,\alpha)\right]$.

(b) If f is differentiable in the first form (a) and f' is differentiable in the 2nd form (b) or if f is differentiable in the 2nd form (b) and f' is differentiable in the first form (a) in definition 2.2 , then then $\underline{f}'(t,\alpha)$ and $\overline{f}'(t,\alpha)$ are differentiable and $f''(t)=\left[\overline{f}''(t,\alpha),\underline{f}''(t,\alpha)\right]$.

Theorem 2.3: Let $f(x)$ be a fuzzy real valued function on $[a,\infty)$ and it is represented by $\left[\underline{f}(x,\alpha),\overline{f}(x,\alpha)\right]$. For any fixed $r\in[0,1]$, assume $\underline{f}(x,\alpha),\overline{f}(x,\alpha)$ are Rimann-integrable on $[a,b]$ for every $b\geq a$ and let there exists two positive $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ such that $\int_a^b\left|\underline{f}(x,\alpha)\right|dx\leq\underline{M}(\alpha)$ and $\int_a^b\left|\overline{f}(x,\alpha)\right|dx\leq\overline{M}(\alpha)$ for every $b\geq a$. Then $f(x)$ is improper fuzzy Rimann-integrable on $[a,\infty)$ and the improper fuzzy Rimann-integral is a fuzzy number. Furthermore, we have –

$$\int_a^\infty f(x)dx=\left(\int_a^\infty\underline{f}(x,\alpha)dx,\int_a^\infty\overline{f}(x,\alpha)dx\right)$$

Proposition2.1: Let $f(x)$ and $g(x)$ be a fuzzy real valued function and also fuzzy Rimann-integrable on $I=[a,\infty)$, then $f(x)+g(x)$ is Rimann-integrable on $I=[a,\infty)$ and,

$$\int_I[f(x)+g(x)]dx=\int_I f(x)dx+\int_I g(x)dx$$

Definition 2.3: The function $u:(a,b)\times(a,b)\to L(R)$ is said to be H- differentiable of the *n*th order at $t_0\in(a,b)$ w.r.t t, if there exists an element $\frac{\partial^n}{\partial t^n}u(x,t_0)\in L(R)$ such that

for all $h>0$ sufficiently small there exists $\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0+h)-\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0)$, $\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0)-\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0-h)$, the following limit hold –

$$\lim_{h\to0}\frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0+h)-\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0)}{h}=\lim_{h\to0}\frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0)-\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0-h)}{h}=\frac{\partial^n}{\partial t^n}u(x,t_0)$$

or

$$\lim_{h\to 0}\frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0)-\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0+h)}{-h}=\lim_{h\to 0}\frac{\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0-h)-\frac{\partial^{n-1}}{\partial t^{n-1}}u(x,t_0)}{-h}=\frac{\partial^{n}}{\partial t^{n}}u(x,t_0)$$

Definition 2.4: Similarly, The function $u:(a,b)\times(a,b)\to L(R)$ is said to be H- differentiable of the *n*th order at $x_0\in(a,b)$ w.r.t x, if there exists an element $\frac{\partial^{n}}{\partial t^{n}}u(x_0,t)\in L(R)$ such that for all $h>0$ sufficiently small there exists $\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h,t)-\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0,t)$, $\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0,t)-\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h,t)$, the following limit hold –

$$\lim_{h\to 0}\frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h,t)-\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0,t)}{h}=\lim_{h\to 0}\frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0,t)-\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h,t)}{h}=\frac{\partial^{n}}{\partial x^{n}}u(x_0,t)$$

or

$$\lim_{h\to 0}\frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0,t)-\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0+h,t)}{-h}=\lim_{h\to 0}\frac{\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0-h,t)-\frac{\partial^{n-1}}{\partial x^{n-1}}u(x_0,t)}{-h}=\frac{\partial^{n}}{\partial x^{n}}u(x_0,t)$$

Definition 2.5: Two Dimensional Fuzzy Laplace Transformation

Let $u=u(x,t)$ is a fuzzy valued function and *s* be real parameter. The fuzzy Laplace transform of the fuzzy real valued function u denoted by $U(x,s)$ is defined as follows:

$$U(x,s)=L[u(x,t)]=\int_0^\infty e^{-st}u(x,t)dt=\lim_{\tau\to\infty}\int_0^\tau e^{-st}u(x,t)dt,$$

$$U(x,s)=\left[\lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{u}(x,t)dt,\lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{u}(x,t)dt\right],\text{ if the limits exist.}$$

The $\alpha-$cut representation of $U(x,s)$ is given by-

$$l[\underline{u}(x,t;\alpha)]=\int_0^\infty e^{-st}\underline{u}(x,t;\alpha)dt=\lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{u}(x,t;\alpha)dt$$

and $$l[\overline{u}(x,t;\alpha)]=\int_0^\infty e^{-st}\overline{u}(x,t;\alpha)dt=\lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{u}(x,t;\alpha)dt$$

Theorem 2.4: Let $u:(a,b)\times(a,b)\to L(R)$ is a fuzzy valued function such that its derivatives up to (n-1)th order w.r.t 't' are continuous for all $t>0$ and u^n exists then-

$$L\left[\frac{\partial^n}{\partial x^n}u(x,t)\right]=s^nU(x,s)-s^{n-1}u(x,0)-s^{n-2}u^1(x,0)-.......-u^{n-1}(x,0)$$

$$L\left[\frac{\partial^n u}{\partial x^n}\right]=\frac{d^n}{dx^n}L\left[u(x,t)\right]=\frac{d^n}{dx^n}U(x,s)$$

Theorem 2.4: **Fuzzy Convolution Theorem**

If f and g are piecewise continuous fuzzy real valued function on $[0,\infty)$ with exponential order p, then-

$$L\{(f*g)(t)\}=L\{f(t)\}.L\{g(t)\}=F(s).G(s)\ ,\ \ s>p\ .$$

3. MAIN METHOD

Consider the following one dimensional Fuzzy wave equation:

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\quad .... \qquad (3.1)$$

with initial condition $u(x,0)=f(x)=\left(\underline{f}(x),\overline{f}(x)\right)$; $u_t(x,0)=g(x)=\left(\underline{g}(x),g(x)\right)$

and boundary condition $u(0,t)=p(t)=\left(\underline{p}(t),\overline{p}(t)\right)$; $u_x(0,t)=q(t)=\left(\underline{q}(t),\overline{q}(t)\right)$.

Taking Fuzzy Laplace Transformation on both sides of (3.1) , we get –

$$\frac{d^2U}{dx^2}=\frac{1}{c^2}\left\{s^2U-su(x,0)-u_t(x,0)\right\}\ ...... \qquad (3.2)$$

Applying Fuzzy Laplace Transformation, boundary condition becomes –

$$U(0,s)=P(s)\ ;\ U_x(0,s)=Q(s)$$

The $\alpha-$cut representation of (3.2) after using initial conditions be given as follows:

$$\frac{d^2\underline{U}}{dx^2}=\frac{1}{c^2}\left\{s^2\underline{U}-s\underline{f}(x)-\underline{g}(x)\right\}\ .... \qquad (3.3)$$

and
$$\frac{d^2\overline{U}}{dx^2}=\frac{1}{c^2}\left\{s^2\overline{U}-s\overline{f}(x)-\overline{g}(x)\right\}\ .... \qquad (3.4)$$

The equation (3.3) is a 2nd order ordinary differential equation which gives the value of $\underline{U}$ and applying boundary conditions then after taking inverse Laplace Transformation, we can obtain the lower solution $\underline{u}(x,t)$ of (3.1). Similarly upper solution $\overline{u}(x,t)$ from (3.4).

4. NUMERICAL PROBLEMS

Example 4.1: Consider the one dimensional fuzzy wave equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\ldots\ldots \qquad (4.1.1)$$

where $u(x,t)$ is the deflection of the string. Let it be stretched between two points (0,0) and (a,0). We are to find $u(x,t)$ under the initial condition and boundary condition:

$$u(x,0)=(1+\alpha,3-\alpha)x\ ;\ u_t(x,0)=(\alpha-1,1-\alpha)e^x\quad ,\ 0\le\alpha\le 1.$$

and

$$u(0,t)=(0,0)\ ;\ u(a,t)=(0,0)$$

Taking Fuzzy Laplace Transformation on both sides of (4.1.1) , we get –

$$\frac{d^2U}{dx^2}=\frac{1}{c^2}\left[s^2U-su(x,0)-u_t(x,0)\right]\ldots. \qquad (4.1.2)$$

Applying Fuzzy Laplace Transformation, boundary condition becomes –

$$U(0,s)=(0,0)\ ;\ U_x(a,s)=(0,0).$$

The $\alpha-$cut representation of (4.2) after using initial conditions be given as follows:

$$\frac{d^2\underline{U}}{dx^2}=\frac{1}{c^2}\left[s^2\underline{U}-s(1+\alpha)x-(\alpha-1)e^x\right]\ldots\ldots\ldots\ldots. \qquad (4.1.3)$$

and $$\frac{d^2\overline{U}}{dx^2}=\frac{1}{c^2}\left[s^2\overline{U}-s(3-\alpha)x-(1-\alpha)e^x\right]\ldots\ldots\ldots\ldots. \qquad (4.1.4)$$

Solving (4.1.3) we get –

$$\underline{U}=Ae^{\frac{sx}{c}}+Be^{-\frac{sx}{c}}+\frac{(1+\alpha)c^2x}{s}+\frac{(\alpha-1)c^2e^x}{s^2-c^2}\ldots\ldots.. \qquad (4.1.5)$$

After using boundary condition we obtained from (4.1.5)

$$A=\frac{1}{\left(e^{-\frac{sa}{c}}-e^{\frac{sa}{c}}\right)}\left[-\frac{(\alpha-1)c^2e^{-\frac{sa}{c}}}{s^2-c^2}+\frac{(1+\alpha)c^2a}{s}-\frac{(\alpha-1)c^2e^a}{s^2-c^2}\right]$$

and

$$B=-\frac{1}{\left(e^{-\frac{sa}{c}}-e^{\frac{sa}{c}}\right)}\left[-\frac{(\alpha-1)c^2e^{\frac{sa}{c}}}{s^2-c^2}+\frac{(1+\alpha)c^2a}{s}-\frac{(\alpha-1)c^2e^a}{s^2-c^2}\right]$$

Taking inverse Laplace Transformation on both sides of (4.1.5), we get –

$$\underline{u}(x,t)=L^{-1}\left[Ae^{\frac{sx}{c}}+Be^{-\frac{sx}{c}}+\frac{(1+\alpha)c^2x}{s}+\frac{(\alpha-1)c^2e^x}{s^2-c^2}\right]\ldots.. \qquad (4.1.6)$$

Similarly from (4.4) we can find that –

$$\overline{u}(x,t)=L^{-1}\left[Ae^{\frac{sx}{c}}+Be^{-\frac{sx}{c}}+\frac{(3-\alpha)c^2x}{s}+\frac{(1-\alpha)c^2e^x}{s^2-c^2}\right].,\ldots \qquad (4.1.7)$$

$$,0\le\alpha\le1.$$

Equations (4.1.6) and (4.1.7) give the lower and upper solutions of (4.1.1) respectively.

Example 4.2: To solve the following equation:

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial^2u}{\partial t^2}\quad\ldots\ldots.. \qquad (4.2.1)$$

with initial condition $u(x,0)=(\alpha+1,3-\alpha)x$; $u_t(x,0)=(\alpha,2-\alpha)$

and boundary condition where $u(0,t)=(0,0)$; $u_x(0,t)=(0,0)$. $\quad 0\le\alpha\le1$.

Taking Fuzzy Laplace Transformation on both sides of (4.2.1) , we get –

$$\frac{d^2U}{dx^2}=s^2U-su(x,0)-u_t(x,0)\ldots\ldots \qquad (4.2.2)$$

Applying Fuzzy Laplace Transformation, boundary condition becomes –

$$U(0,s)=(0,0)\ ;\ U_x(0,s)=(0,0).$$

The $\alpha-$cut representation of (4.2.2) after using initial conditions be given as follows:

$$\frac{d^2\underline{U}}{dx^2} = s^2\underline{U} - s(\alpha+1)x - \alpha \text{ ....} \qquad (4.2.3)$$

and $$\frac{d^2\overline{U}}{dx^2} = s^2\overline{U} - s(3-\alpha)x - (2-\alpha) \text{ ....} \qquad (4.2.4)$$

Solving (4.3) we get –

$$\underline{U} = Ae^{sx} + Be^{-sx} + (\alpha+1)\frac{x}{s} + \frac{\alpha}{s^2} \text{........} \qquad (4.2.5)$$

After using boundary condition we obtained from (4.2.5)

$$\underline{U} = -\frac{(2\alpha+1)}{2s^2}e^{sx} + \frac{1}{2s^2}e^{-sx} + (\alpha+1)\frac{x}{s} + \frac{\alpha}{s^2} \text{........} (4.2.6)$$

Taking inverse Laplace Transformation, we get –

$$\underline{u}(x,t) = -\left(\alpha+\frac{1}{2}\right)L^{-1}\left[\frac{e^{sx}}{s^2}\right] + \frac{1}{2}L^{-}\left[\frac{e^{-sx}}{s^2}\right] + (\alpha+1)x + \frac{\alpha}{2}t\text{....} \qquad (4.2.7)$$

$$,\ 0 \le \alpha \le 1.$$

Similarly from (4.2.4) we can find that –

$$\overline{u}(x,t) = -\left(\frac{5}{2}-\alpha\right)L^{-1}\left[\frac{e^{sx}}{s^2}\right] + \frac{1}{2}L^{-1}\left[\frac{e^{-sx}}{s^2}\right] + (3-\alpha)x + (2-\alpha)\frac{t}{2}\text{...} \qquad (4.2.8)$$

$$,\ 0 \le \alpha \le 1.$$

Equations (4.2.7) and (4.2.8) give the lower and upper solutions of (4.2.1) respectively.

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